THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3070 (Second Term, 2016–2017)
Introduction to Topology
Exercise 9a Compactness

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Let $(X, \mathfrak{T})$ be given the cofinite topology.
   (a) Show that X is compact.
   (b) Show that every subset $A \subset X$ is compact.
   (c) Do you think the same hold for co-countable topology?
2. A family $\mathcal{F}$ of closed sets satisfies *finite intersection property* if every intersection of finitely many sets in $\mathcal{F}$ is nonempty. Prove that the following is equivalent to compactness: every family $\mathcal{F}$ of closed sets satisfying the finite intersection property must have $\cap\mathcal{F}$ nonempty.

   A family $\mathcal{C}$ of sets (not necessarily open nor closed) satisfies *finite closure intersection property* if for each finite $\mathcal{A} \subset \mathcal{C}$, the intersection $\cap \left\{ \overline{A} : A \in \mathcal{A} \right\} \neq \emptyset$. Show that compactness is equivalent to: every family $\mathcal{C}$ satisfying the finite closure intersection property must have $\cap \left\{ \overline{C} : C \in \mathcal{C} \right\} \neq \emptyset$.
3. Let $\mathcal{B}$ be a base for $\mathfrak{T}$. Assume that every open cover $\mathcal{C} \subset \mathcal{B}$ for X has a finite subcover. Prove that X is compact. *Remark.* The converse is trivially true.

   Remark. The same question concerning subbase is considerably harder.
4. Show that if a space $(X, \mathfrak{T})$ is compact and discrete then X is finite.
5. Use the indiscrete topology to create an example of a compact space X with a compact subset A which is not closed in X.
6. Let X be compact and $F_n \subset X$ be nonempty closed sets such that $F_{n+1} \subset F_n$ for each $n \in \mathbb{N}$. Show that $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.
7. Recall that a set S in a metric space (Y, d) is *bounded* iff $S \subset B(y_0, R)$ for some $y_0 \in Y$ and $R > 0$. Let (X, d) be a metric space. Prove that if $K \subset X$ is compact, it is closed and bounded.

   Do you think the converse is true?

8. Prove that if $(X, \mathfrak{T})$ is compact and $f\colon (X, \mathfrak{T}) \to (Y, d)$ is continuous, then the image $f(X)$ is bounded.

9. Let K_α be compact subsets in a topological space $(X, \mathfrak{T})$. Prove that a finite union of K_α's is compact and, if X is Hausdorff, an arbitrary intersection of K_α's is compact.

   Think about what happens to infinite union of compact sets.

10. Let $C(X) = \{ f\colon X \to \mathbb{R} \mid f \text{ is continuous} \}$. Prove that if X is compact, then

$$d(f, g) = \sup \{ |f(x) - g(x)| \ : \ x \in X \}$$

   defines a metric on $C(X)$.